HYPERSURFACES OF ODD-DIMENSIONAL SPHERES

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A structure similar to an almost complex structure was shown in [2] to exist on a hypersurface of an almost contact manifold or a submanifold of codimension 2 of an almost complex space. This structure on a manifold M has been studied in [1], [5], [6] from two points of view, namely, that the structure exists on M because M is a submanifold of some ambient space, and also that the structure exists intrinsically on M.

The odd-dimensional sphere S^{2n+1} has an almost contact structure which is naturally induced from the Kaehler structure of Euclidean space E^{2n+2}. The purpose of this paper is to study complete hypersurfaces immersed in S^{2n+1}. In § 3 it is shown that if the Weingarten map of the immersion and f commute then the hypersurface is a sphere whose radius is determined. Here, f is a tensor field of type $(1,1)$ on the hypersurface, which is part of the induced structure. That the hypersurface satisfying this condition is a sphere follows from the results in [6], however a new proof is given here for completeness. In § 4 it is shown that if the Weingarten map K of the immersion and f satisfy $fK + Kf = 0$, and the hypersurface is of constant scalar curvature, then it is a great sphere or $S^n \times S^n$.

1. Hypersurfaces of a sphere

Let S^{2n+1} be the natural sphere of dimension $2n + 1$ in Euclidean $(2n + 2)$-space E^{2n+2}. Let (ϕ, ξ, η, g) be the normal, almost contact metric structure (see [4]) induced on S^{2n+1} by the Kaehler structure on E^{2n+2}. That is to say, ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on S^{2n+1} satisfying

$$\phi^2 = -I + \eta \otimes \xi ,$$
$$\phi\xi = 0 , \qquad \eta \circ \phi = 0 ,$$
(1)
$$\eta(\xi) = 1 ,$$
$$g(\phi\bar{X}, \phi\bar{Y}) + \eta(\bar{X})\eta(\bar{Y}) = g(\bar{X}, \bar{Y}) ,$$
$$[\phi, \phi] + d\eta \otimes \xi = 0 ,$$

where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ, and $\bar{X}$ and $\bar{Y}$ are arbitrary vector fields on S^{2n+1}.

Suppose $\pi\colon M^{2n} \to S^{2n+1}$ is an embedding of the orientable manifold M^{2n} in S^{2n+1}. The tensor G defined on M^{2n} by

$$(2) \qquad G(X, Y) = g(\pi_* X, \pi_* Y)$$

is a Riemannian metric on M^{2n}, where π_* denotes the differential of the embedding π. If C is a field of unit normals defined on M^{2n}, and $\tilde{V}$ is the Riemannian connection of g, then the Gauss and Weingarten equations can be written as

$$(3) \qquad \begin{aligned} \tilde{V}_{\pi_* X} \pi_* Y &= \pi_*(V_X Y) + k(X, Y)C , \\ \tilde{V}_{\pi_* X} C &= \pi_*(KX) . \end{aligned}$$

Then V is the Riemannian connection of G, k is a symmetric tensor of type $(0,2)$ on M^{2n}, and $G(KX, Y) = k(X, Y)$. Furthermore, if we set

$$(4) \qquad \begin{aligned} \phi \pi_* X &= \pi_* f X + v(X)C , & \xi &= \pi_* U + \lambda C , \\ \phi C &= -BV , & u(X) &= \eta(\pi_* X) , \end{aligned}$$

then f is a tensor field of type $(1,1)$, U and V are vector fields, u and v are 1-forms, and λ is a function satisfying

$$(5) \qquad \begin{aligned} f^2 &= -I + u \otimes U + v \otimes V , \\ u \circ f &= \lambda v , & v \circ f &= -\lambda u , \\ f U &= -\lambda V , & f V &= \lambda U , \\ u(U) = v(V) &= 1 - \lambda^2 , & u(V) = v(U) &= 0 , \\ G(fX, fY) &= G(X, Y) - u(X)u(Y) - v(X)v(Y) . \end{aligned}$$

It was shown in [2] that the following relations hold

$$(6) \qquad \begin{aligned} (V_X f)Y &= G(X, Y)U - u(Y)X - k(X, Y)V + v(Y)KX , \\ V_X U &= -fX - \lambda KX , \\ V_X V &= -\lambda X + fKX , \\ V_X \lambda &= v(X) + k(U, X) . \end{aligned}$$

2.   Case I: $Kf - fK = 0$

We will prove the following theorem.

Theorem 1.   *If M^{2n} is an orientable submanifold of S^{2n+1} satisfying $Kf = fK$, and $\lambda \neq$ constant, K being the Weingarten map of the embedding, and f and λ being defined in (4), then M^{2n} is a sphere of radius $1/\sqrt{1 + \alpha^2}$, where α is some constant determined by the embedding.*

Proof. We have that $0 = G((Kf - fK)U, U)$, so that

$$0 = G(KfU, U) - G(fKU, U)$$
$$= -\lambda G(V, KU) + G(KU, fU)$$
$$= -\lambda G(V, KU) - \lambda G(KU, V) .$$

Therefore we see $\lambda = 0$ or $k(U, V) = 0$. By continuity, since λ is non-constant,

(7) $$k(U, V) = 0 .$$

In a similar fashion we obtain

(8) $$k(U, U) = k(V, V) .$$

Now $fKU + \lambda KV = 0$, so that

$$0 = -KU + u(KU)U + v(KU)V + \lambda KfV$$
$$= -KU + u(KU)U + \lambda^2 KU ,$$

and hence

$$(1 - \lambda^2)KU = k(U, U)U .$$

Similarly, we obtain

$$(1 - \lambda^2)KV = k(U, U)V .$$

At points where $\lambda \neq \pm 1$, we have $KU = \alpha U$ for $\alpha = k(U, U)/(1 - \lambda^2)$, which implies

$$(\nabla_X K)U + K(-fX - \lambda KX) = \nabla_X \alpha \cdot U + \alpha(-fX - \lambda KX) .$$

The Codazzi equation for an embedding gives that $(\nabla_X K)(Y) = (\nabla_Y K)(X)$ so that we have

$$2G(KfZ, X) = (\nabla_X \alpha)u(Z) - (\nabla_Z \alpha)u(X) + 2\alpha G(fZ, X) .$$

If we set Z equal to U, then

$$-2\alpha \lambda u(X) = (\nabla_X \alpha)(1 - \lambda^2) - (\nabla_U \alpha)u(X) - 2\lambda u(X) ,$$

so that $\nabla_X \alpha$ and $u(X)$ are proportional. Therefore $Kf = \alpha f$, and hence

$$-KX + u(X)KU + v(X)KV = \alpha(-X + u(X)U + v(X)V) .$$

Thus $KX = \alpha X$ for all X, and by the Codazzi equation α is constant. From (6) we have that $\nabla_X \lambda = v(X) + \alpha u(X)$, and therefore that

$$\nabla_Y \nabla_X \lambda - d\lambda(\nabla_Y X) = \nabla_Y(v(X) + \alpha u(X)) - (v(\nabla_Y X) + \alpha u(\nabla_Y X))$$
$$= -\lambda G(X, Y) + \alpha G(X, fY) - \alpha G(X, fY) - \alpha \lambda G(KY, X)$$
$$= -\lambda(1 + \alpha^2)G(X, Y) .$$

By the following lemma of Obata [3], M^{2n} is sphere of radius $(1 + \alpha^2)^{-1/2}$.

Lemma. *A complete connected Riemannian manifold M admits a nontrivial solution λ of $\nabla_Y \nabla_X \lambda - d\lambda(\nabla_Y X) = -k\lambda G(X, Y)$ for some real number $k > 0$ if and only if M is globally isometric to a Euclidean sphere of radius $k^{-1/2}$.*

Corollary. *Let M^{2n} be an orientable submanifold of S^{2n+1} with $\lambda \neq$ constant. Then $Kf - fK = 0$ if and only if M^{2n} is a totally umbilical submanifold of S^{2n+1}.*

Remark. In [5], there was introduced the idea of *normality* of an (f, G, u, v, λ)-structure, which is of a manifold M^{2n} with tensors satisfying (5). This condition is

$$[f, f] + du \otimes U + dv \otimes V = 0 .$$

We have the following proposition.

Proposition. *Let M^{2n} be a hypersurface of S^{2n+1} with $\lambda \neq$ constant. The (f, G, u, v, λ)-structure on M^{2n} is normal if and only if $fK - Kf = 0$.*

Proof. Let

$$S(X, Y) = [f, f](X, Y) + du(X, Y)U + dv(X, Y)V .$$

Using (5) it can be shown that

$$S(X, Y) = v(Y)(Kf - fK)X - v(X)(Kf - fK)Y ,$$

and hence the "if" part of the proposition is proved. On the other hand, assume $S(X, Y) = 0$ for all X and Y and let $PX = (Kf - fK)X$. Then

$$v(V)PX = v(X)PV .$$

Also, it can be shown that

$$G(PX, Y) = G(X, PY)$$

so that

$$v(X)G(PV, Y) = v(Y)G(PV, X) ,$$

that is to say,

$$G(PV, Y) = \alpha v(Y)$$

for some α. Thus we have that

$$v(V)G(PX, Y) = v(X)G(PV, Y) = \alpha v(X)v(Y) ,$$

but since the trace of P is 0, we have $\alpha = 0$ and thus $P = 0$.

3. Case II: $Kf + fK = 0$

In this section we prove the following theorem.

Theorem 2. If M^{2n} is a complete orientable submanifold of S^{2n+1} with constant scalar curvature satisfying $Kf + fK = 0$ and $\lambda \neq$ constant, where K is the Weingarten map of the embedding, and f and λ are defined in (4), then M^{2n} is a natural sphere S^{2n} or $M^{2n} = S^n \times S^n$.

Proof. We have that

$$0 = (Kf + fK)U = -\lambda KV + fKU ,$$
$$0 = (Kf + fK)V = \lambda KU + fKV ,$$

so that

$$\begin{aligned} 0 &= -\lambda k(V, V) + G(fKU, V) \\ &= -\lambda k(V, V) - G(KU, fV) \\ &= -\lambda k(V, V) - \lambda k(U, U) , \end{aligned}$$

and hence

$$(9) \qquad\qquad k(U, U) + k(V, V) = 0$$

by continuity. Also

$$\begin{aligned} 0 &= -\lambda fKV + f^2KU \\ &= \lambda^2 KU + (-KU + u(KU)U + v(KU)V) , \end{aligned}$$

that is,

$$(10) \qquad\qquad (1 - \lambda^2)KU = k(U, U)U + k(U, V)V ,$$

and similarly

$$(11) \qquad\qquad (1 - \lambda^2)KV = k(U, V)U + k(V, V)V .$$

At points where $\lambda \neq \pm 1$, write equations (10) and (11) as

$$(10') \qquad\qquad KU = \alpha U + \beta V ,$$
$$(11') \qquad\qquad KV = \beta U - \alpha V .$$

If we apply ∇_X to equation (10'), use equation (6) for $\nabla_X U$ and $\nabla_X V$, and use

the fact that $(\nabla_X K)Y = (\nabla_Y K)X$ because of the Codazzi equation, then we find that

$$(12) \qquad \begin{aligned} (\nabla_X \alpha)u(Y) - (\nabla_Y \alpha)u(X) + (\nabla_X \beta)v(Y) \\ - (\nabla_Y \beta)v(X) - 2\alpha F(X, Y) = 0 , \end{aligned}$$

where $F(X, Y) = G(fX, Y)$. Setting $X = U$ and $Y = V$ and using the fact that $\lambda \neq$ constant we see that

$$(13) \qquad -\nabla_V \alpha + \nabla_U \beta + 2\alpha\lambda = 0 .$$

From equations (12) and (13) we obtain

$$(14) \qquad (1 - \lambda^2)\nabla_Y \alpha = (\nabla_U \alpha)u(Y) + (\nabla_V \alpha)v(Y) ,$$

$$(15) \qquad (1 - \lambda^2)\nabla_Y \beta = (\nabla_U \beta)u(Y) + (\nabla_V \beta)v(Y) ,$$

$$(16) \qquad 2\alpha(1 - \lambda^2)F(X, Y) = (u(Y)v(X) - v(Y)u(X))(\nabla_V \alpha - \nabla_U \beta) .$$

However, since the rank of f is $\geq 2n - 2$, equation (16) implies that $\alpha = 0$ and $\nabla_U \beta = 0$ if $n \neq 1$. Thus equation (12) becomes

$$(12') \qquad (\nabla_X \beta)v(Y) = (\nabla_Y \beta)v(X) ,$$

or

$$(12'') \qquad (1 - \lambda^2)\nabla_X \beta = (\nabla_Y \beta)v(X) .$$

Applying ∇_X to equation (11'), and using the fact that $\alpha = 0$ and the Codazzi equation, we find that

$$(17) \qquad (\nabla_X \beta)u(X) - (\nabla_Y \beta)u(X) - 2\beta F(X, Y) - 2F(KX, KY) = 0 .$$

Setting $Y = U$ and using (12'') we have that $2\beta^2\lambda = 2\beta\lambda - \nabla_V \beta$ so that $\beta =$ constant implies that $\beta = 0$ or $\beta = 1$.

Replace Y by fY in equation (17) and use equation (12'') to obtain

$$\begin{aligned} 2(1 - \lambda^2)F(KX, KfY) \\ = (\nabla_V \beta)(v(X)u(fY) - v(fY)u(X)) - 2\beta(1 - \lambda^2)F(X, fY) , \end{aligned}$$

that is,

$$\begin{aligned} -2(1 - \lambda^2)[G(KX, KY) - u(KX)u(KY) - v(KX)v(KY)] \\ = \nabla_V \beta[\lambda v(X)v(Y) + \lambda u(X)u(Y)] \\ - 2\beta(1 - \lambda^2)[G(X, Y) - u(X)u(Y) - v(X)v(Y)] , \end{aligned}$$

from which follows

(18) $(1 - \lambda^2)K^2 = (\beta^2 - \beta)(u \otimes U + v \otimes V) + \beta(1 - \lambda^2)I$.

From (18) and a previous remark we see that if $\beta = $ constant then $K^2 = 0$ or $K^2 = I$. If $K^2 = 0$, then $K = 0$ since K is symmetric. In this case, M^{2n} is a totally geodisic submanifold of S^{2n+1} and hence $M^{2n} = S^{2n}$. In the case where $K^2 = I$, K gives an almost product structure on M^{2n}.

We have

$$k(fX, fY) = G(KfX, fY) = G(Kf^2X, Y)$$
$$= -G(KX, Y) + u(X)G(KU, Y) + v(X)G(KV, Y)$$
$$= -k(X, Y) + \beta(u(X)v(Y) + v(X)u(Y)) .$$

Now since $k(U, U) + k(V, V) = 0$ and $G(U, V) = 0$, the last equation can be used to show that the trace of K is 0, that is, M^{2n} is a minimal hypersurface (note that this last conclusion holds whether or not $\beta = $ constant). In the case where $K^2 = I$, tr $K = 0$ implies that the global distributions on M^{2n} given by $\frac{1}{2}(K + I)$ and $\frac{1}{2}(I - K)$ are both of dimension n.

Now to find the scalar curvature of M^{2n} by the Gauss equation, let $\tilde{R}$ and R denote the curvature tensors of g and G respectively. Then the Gauss equation is

(19)
$$\tilde{R}(\pi_*X, \pi_*Y, \pi_*Z, \pi_*W)$$
$$= R(X, Y, Z, W) - (k(Y, Z)k(X, W) - k(Y, W)k(X, Z)) .$$

Using (18) and the fact that S^{2n+1} is of constant curvature equal to 1, for $1 - \lambda^2 \neq 0$ we have

$$\bar{R}(X, Y) = (2n - 1)g(X, Y)$$
$$- \left[\beta g(X, Y) + \frac{\beta^2 - \beta}{1 - \lambda^2}(u(X)u(Y) + v(X)v(Y)) \right] ,$$

where $\bar{R}$ is the Ricci tensor of G. From this it follows that the scalar curvature of M^{2n} is equal to $2n(2n - 1) - \beta(2n - 2) - 2\beta^2$, and therefore that $\beta = $ constant.

If we apply ∇_X to equation (19) and use the second Bianchi identity and tr $K = 0$, then we obtain that

$$(\nabla_X K)Y + (\nabla_Y K)X = 0 ,$$

and thus $\nabla_X K = 0$ by the Codazzi equation.

Therefore, if $\beta = 1$, the almost product structure K is decomposable. Hence by completeness, M^{2n} is a product $M^n \times \bar{M}^n$. Now we have, by equation (17),

$$(20) \quad \begin{aligned} G(f(K \pm I)X, (K \pm I)Y) \\ = F(KX, KY) + F(X, Y) \pm (F(KX, Y) + F(X, KY)) = 0 \,, \end{aligned}$$

and, by equation (6),

$$\begin{aligned} \nabla_Y \nabla_X \lambda - d\lambda(\nabla_Y(X)) &= \nabla_Y(v(X) + k(U, X)) - (v(\nabla_Y X) + k(U, \nabla_Y X)) \\ &= (\nabla_Y v)X + k(\nabla_Y U, X) = -2\lambda G(X, Y) - 2G(fY, KX) \,. \end{aligned}$$

From equation (20) we see that if X and Y are both in the distribution $I + K$ or $I - K$, then $g(fY, KX) = 0$ so that

$$\nabla_Y \nabla_X \lambda - d\lambda(\nabla_Y X) = -2\lambda G(X, Y) \,.$$

Thus, M^n and $\overline{M}^n$ are both spheres of radius $1/\sqrt{2}$ by the lemma of Obata.

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